Journal of Econometrics and Statistics
Vol. 3, Issue 1, 2023, pp. 37-53
https://doi.org/10.47509/JES.2023.v03i01.03

The odd-log-logistic-Chen distribution: a bimodal distribution with applications

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Abstract

In this paper, an extension of the Chen distribution is defined. The new distribution is based on the odd-log-logistic-G family of distributions, which adds an extra shape parameter to the baseline distribution. Here, the proposed distribution is called odd-log-logistic-Chen. Some properties have been defined. A regression model with parameterization on the median is also proposed. The maximum likelihood method is used to estimate the unknown parameters. The accuracy of the maximum likelihood estimators is shown through Monte Carlo simulations. The usefulness of the new model is shown with three applications to real uncensored data, the odd-log-logistic-Chen distribution being better than other three models known in the literature. From the defined regression model, an application to censored data of laryngeal cancer is also considered.
Keywords: Chen distribution; bimodal distribution; regression model; maximum likelihood; quantile function; Monte Carlo simulation; failure rate function

1 Introduction

In recent years, the Chen (Chen, 2000) distribution used as a baseline for many generators proposed in the literature. Some generators used are exponentiated-G (Dey et al., 2017), Marshall-Olkin-G (Rocha et al., 2017), Kumaraswamy Exponentiated-G (Khan et al., 2018), gamma-G (Reis et al., 2020), among others.

In this paper, the Chen distribution is inserted in odd log-logistic-G (Gleaton and Lynch, 2006) family. Let $G(x;\boldsymbol{\eta})$ be a baseline cumulative distribution function (cdf) indexed by q-vector of parameters $\boldsymbol{\eta}$. The cdf of the odd log-logistic-G (OLL-G) family with one extra shape parameter is given by

$$F_{\text{OLL-G}}(x;a,\boldsymbol{\eta}) = \frac{G(x;\boldsymbol{\eta})^a}{G(x;\boldsymbol{\eta})^a + \bar{G}(x;\boldsymbol{\eta})^a}, \qquad (1)$$

Article History
Received : 28 January 2023; Revised : 26 February 2023; Accepted : 10 March 2023; Published : 27 May 2023

To cite this paper
Lucas David Riberiro-Reis (2023). The odd-log-logistic-Chen distribution: a bimodal distribution with applications. *Journal of Econometrics and Statistics*. 3(1), 37-53. https://doi.org/10.47509/JES.2023.v03i01.03

where $a > 0$ is the extra parameter and $\bar{G}(x;\boldsymbol{\eta}) = 1 - G(x;\boldsymbol{\eta})$. Note that for $a = 1$, the baseline distribution $G(x;\boldsymbol{\eta})$ is obtained. Note that

$$a = \frac{\log[F_{\text{OLL-G}}(x;a,\boldsymbol{\eta})/\bar{F}_{\text{OLL-G}}(x;a,\boldsymbol{\eta})]}{\log[G(x;\boldsymbol{\eta})/\bar{G}(x;\boldsymbol{\eta})]}.$$

Thus, the extra parameter a represents the log-odds ratio between the generated distribution and the baseline distribution.

The corresponding probability density function (pdf) and the hazard rate function (hrf) are

$$f_{\text{OLL-G}}(x;a,\boldsymbol{\eta}) = \frac{a g(x;\boldsymbol{\eta})\{G(x;\boldsymbol{\eta})\bar{G}(x;\boldsymbol{\eta})\}^{a-1}}{\{G(x;\boldsymbol{\eta})^a + \bar{G}(x;\boldsymbol{\eta})^a\}^2} \tag{2}$$

and

$$\varphi_{\text{OLL-G}}(x;a,\boldsymbol{\eta}) = \frac{a g(x;\boldsymbol{\eta}) G(x;\boldsymbol{\eta})^{a-1}}{\bar{G}(x;\boldsymbol{\eta})[G(x;\boldsymbol{\eta})^a + \bar{G}(x;\boldsymbol{\eta})^a]},$$

respectively, where $g(x;\boldsymbol{\eta}) = dG(x;\boldsymbol{\eta})/dx$.

The cdf and pdf of the Chen (Chen, 2000) distribution are given by

$$G(x;\lambda,\beta) = 1 - \mathrm{e}^{\lambda(1-\mathrm{e}^{x^\beta})}, \quad x > 0$$

and

$$g(x;\lambda,\beta) = \lambda\beta x^{\beta-1}\mathrm{e}^{x^\beta+\lambda(1-\mathrm{e}^{x^\beta})}, \quad x > 0, \tag{3}$$

respectively, where, $\lambda > 0$ is the scale parameter and $\beta > 0$ is the shape parameter.

By taking $G(\cdot)$ and $g(\cdot)$ as the cdf and pdf of the Chen distribution, respectively, and substituting in (1) and (2), the cdf and pdf of the OLL-Chen (OLLC) distribution are

$$F_{\text{OLLC}}(x;a,\lambda,\beta) = \frac{\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x^\beta})}\right]^a}{\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x^\beta})}\right]^a + \mathrm{e}^{a\lambda(1-\mathrm{e}^{x^\beta})}}$$

and

$$f_{\text{OLLC}}(x;a,\lambda,\beta) = \frac{a\lambda\beta x^{\beta-1}\mathrm{e}^{x^\beta+\lambda(1-\mathrm{e}^{x^\beta})}\left[\mathrm{e}^{\lambda(1-\mathrm{e}^{x^\beta})} - \mathrm{e}^{2\lambda(1-\mathrm{e}^{x^\beta})}\right]^{a-1}}{\left\{\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x^\beta})}\right]^a + \mathrm{e}^{a\lambda(1-\mathrm{e}^{x^\beta})}\right\}^2}. \tag{4}$$

For $a = 1$, the OLLC distribution is reduced to Chen distribution. The random variable X with pdf (4) is denoted as $X \sim \text{OLLC}(a,\lambda,\beta)$.

The corresponding hrf of X is

$$\varphi_{\text{OLLC}}(x;a,\lambda,\beta) = \frac{a\lambda\beta x^{\beta-1}\mathrm{e}^{x^\beta}\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x^\beta})}\right]^{a-1}}{\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x^\beta})}\right]^a + \mathrm{e}^{a\lambda(1-\mathrm{e}^{x^\beta})}}.$$

Figure 1 shows some forms of OLLC density. Note that, this new pdf supports several forms, including one of the most sought after in the literature, which is the bimodality. Figure 2 shows some plots of the hrfs of X. The hrf can take various forms, such as: decreasing, unimodal, bathtub, unimodal-bathtub. This shows the flexibility that the additional parameter a has.

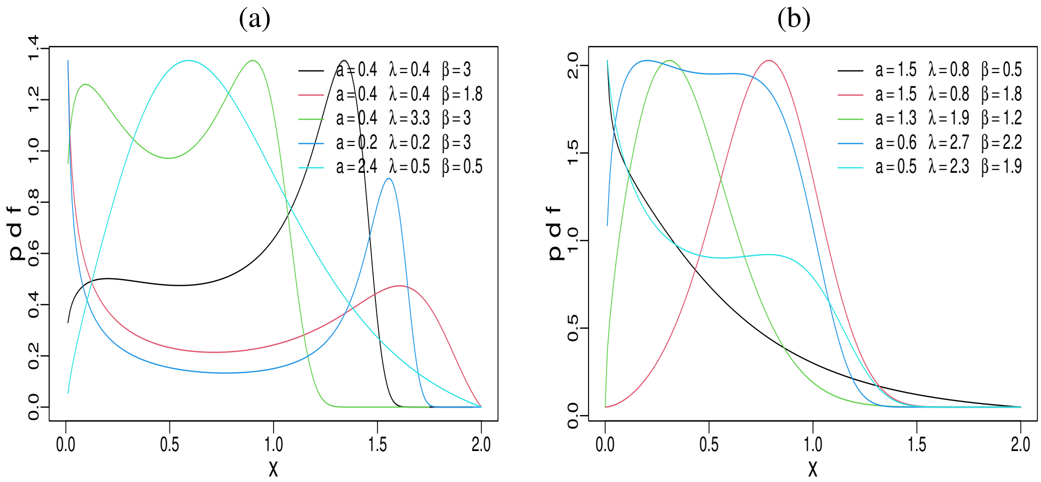

Figure 1: Some OLLC pdfs.

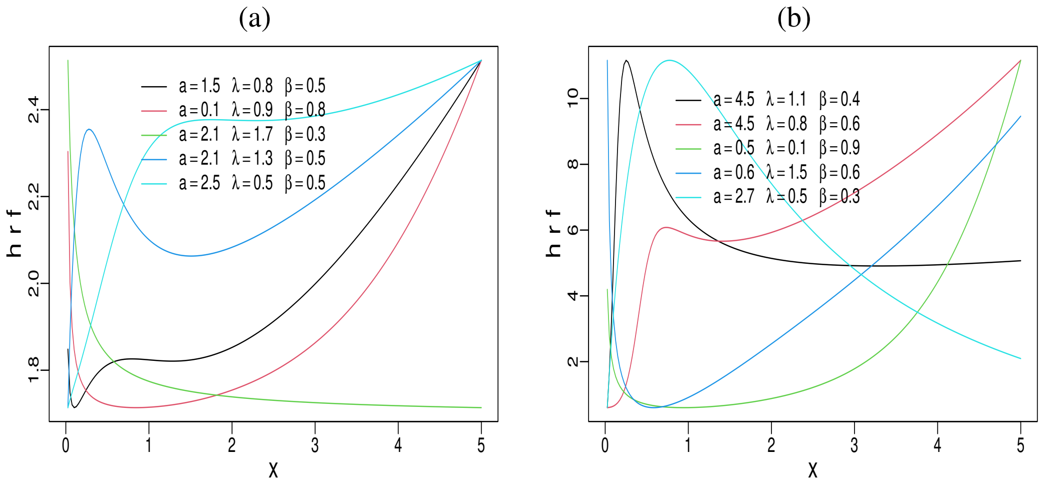

Figure 2: Some OLLC hrfs.

This paper is organized as follows. In Sections 2 and 3, the linear combination and some properties of the OLLC distribution are described, respectively. The maximum likelihood estimation method and Monte Carlo simulations are presented in Section 4. In section 5, a regression model is introduced. Maximum likelihood estimation is discussed and Monte Carlo simulations are performed. In Section 6, the usefulness of the proposed model is shown through applications to uncensored and censored data. Finally, Section 7 concludes the paper.

2 Expansion for density

For a given cdf $G(z;\boldsymbol{\eta})$ with parameter q-vector $\boldsymbol{\eta}$, the random variable Z is called of exp-G with power parameter $a>0$, if its cdf and pdf are

$$H_{\text{EG}}(z;a,\boldsymbol{\eta}) = G(z;\boldsymbol{\eta})^{a} \quad \text{and} \quad h_{\text{EG}}(z;a,\boldsymbol{\eta}) = a\,g(z;\boldsymbol{\eta})G(z;\boldsymbol{\eta})^{a-1},$$

respectively, where $g(z;\boldsymbol{\eta}) = dG(z;\boldsymbol{\eta})/dz$. The random variable Z is denoted as $Z \sim$ exp-G$(a,\boldsymbol{\eta})$.

Using the generalized binomial expansions, Cordeiro et al. (2016) showed that the Equation (2) can be written as

$$f_{\text{OLL-G}}(x;a,\boldsymbol{\eta}) = \sum_{k=0}^{\infty} w_k\, h_{\text{EG}}(x;(k+1),\boldsymbol{\eta}), \tag{5}$$

where $h_{\text{EG}}(x;(k+1),\boldsymbol{\eta})$ is the exp-G$((k+1),\boldsymbol{\eta})$ pdf and the coefficients w_k are

$$w_k = w_k(a) = \frac{a}{(k+1)} \sum_{i,j=0}^{\infty} \sum_{l=k}^{\infty} (-1)^{j+k+l} \binom{-2}{i} \binom{-a(i+1)}{j} \binom{a(i+1)+j-1}{l} \binom{l}{k}.$$

Which shows that the OLL-G family pdf is a linear combination of exp-G densities. By integration corresponding cdf is

$$F_{\text{OLL-G}}(x;a,\boldsymbol{\eta}) = \sum_{k=0}^{\infty} w_k\, H_{\text{EG}}(x;(k+1),\boldsymbol{\eta}),$$

where $H_{\text{EG}}(x;(k+1),\boldsymbol{\eta})$ is the exp-G$((k+1),\boldsymbol{\eta})$ cdf.

According to Reis et al. (2020), the pdf of the exp-Chen(a,λ,β) distribution can be expressed as

$$h_{\text{EC}}(x;a,\lambda,\beta) = \sum_{m=1}^{\infty} (-1)^{m+1} \binom{a}{m} g(x;m\lambda,\beta), \tag{6}$$

where $g(x;m\lambda,\beta)$ is the Chen density function with scale parameter $m\lambda$ and shape parameter β. So, the exp-Chen pdf can be write as a linear combination of Chen densities.

Then, inserting (6) in (5), the pdf (4) can be written as

$$f_{\text{OLLC}}(x;a,\lambda,\beta) = \sum_{m=1}^{\infty} v_m\, g(x;m\lambda,\beta), \tag{7}$$

where

$$v_m = v_m(a) = (-1)^{m+1} \sum_{k=0}^{\infty} w_k \binom{k+1}{m}.$$

So, is demonstrated that the pdf of the OLLC distribution can be expressed as a mixture of Chen densities. The corresponding cdf is obtained by integration

$$F_{\text{OLLC}}(x;a,\lambda,\beta) = \sum_{m=1}^{\infty} v_m\, G(x;m\lambda,\beta),$$

where $G(x;m\lambda,\beta)$ is the Chen$(m\lambda,\beta)$ cdf.

3 Properties

The quantile function (qf) of the OLL-G family, say $Q_{\text{OLL-G}}(p;a,\boldsymbol{\eta}) = F^{-1}_{\text{OLL-G}}(p;a,\boldsymbol{\eta})$, is given by

$$Q_{\text{OLL-G}}(p;a,\boldsymbol{\eta}) = Q_{\text{G}}\left(\frac{p^{1/a}}{p^{1/a}+(1-p)^{1/a}};\boldsymbol{\eta}\right), \quad 0 < p < 1,$$

where Q_{G} is the qf of the baseline $G(x;\boldsymbol{\eta})$.

Then, the qf of the OLLC distribution becomes

$$Q_{\mathrm{OLLC}}(p;a,\lambda,\beta)=\left\{\log\left[1-\lambda^{-1}\log\left(\frac{(1-p)^{1/a}}{p^{1/a}+(1-p)^{1/a}}\right)\right]\right\}^{1/\beta},\quad 0<p<1. \tag{8}$$

When $p=0.5$, the median of the OLLC distribution is obtained. Thus, the median of X is

$$\mathrm{median}(X)=\left\{\log\left[1-\lambda^{-1}\log(0.5)\right]\right\}^{1/\beta}.$$

Note that the median of the proposed distribution does not depend on the additional parameter a. Table 1 presents the median of X for selected values of λ and β. Note that for fixed β, when λ increases, the median decreases. On the other hand, fixing λ, when β increases, the median also increases.

Table 1: Median of X for selected values of λ and β.

	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$
$\lambda=1$	0.5265	0.7256	0.8075	0.8519
$\lambda=2$	0,2975	0.5454	0.6676	0.7385
$\lambda=3$	0.2078	0.4559	0.5923	0.6752
$\lambda=4$	0.1598	0.3998	0.5427	0.6323

From Equation (8), X can be easily simulated. Basically, if U is a random variable uniformly distributed on unit interval, say $U\sim\mathcal{U}(0,1)$, then the random variable

$$X=\left\{\log\left[1-\lambda^{-1}\log\left(\frac{(1-U)^{1/a}}{U^{1/a}+(1-U)^{1/a}}\right)\right]\right\}^{1/\beta} \tag{9}$$

has pdf (4). Using (9), some values of X was simulated and the histograms of these values are displayed in Figure 3. The red line represents the respective pdf for the same parameters as the values were generated. It is noted the great flexibility that this new distribution has.

From Equation (8), expressions for skewness and kurtosis can be obtained. The Bowley skewness is based on quartiles and the Moors kurtosis is based on octiles. Let $Q_{\mathrm{OLLC}}(p)=Q_{\mathrm{OLLC}}(p;a,\lambda,\beta)$, the expressions for this skewness and kurtosis of X are given by

$$\mathcal{B}(a,\lambda,\beta)=\frac{Q_{\mathrm{OLLC}}(3/4)+Q_{\mathrm{OLLC}}(1/4)-2Q_{\mathrm{OLLC}}(2/4)}{Q_{\mathrm{OLLC}}(3/4)-Q_{\mathrm{OLLC}}(1/4)}$$

and

$$\mathcal{M}(a,\lambda,\beta)=\frac{Q_{\mathrm{OLLC}}(7/8)-Q_{\mathrm{OLLC}}(5/8)-Q_{\mathrm{OLLC}}(3/8)+Q_{\mathrm{OLLC}}(1/8)}{Q_{\mathrm{OLLC}}(6/8)-Q_{\mathrm{OLLC}}(2/8)},$$

respectively. Plots of the skewness and kurtosis of X as function of a for selected values of λ and β are displayed in Figure 4. This Figure show that for $\beta<2.5$, the skewness has a decreasing behavior when the parameter a increases. On the other hand, for $\beta>2.5$, the skewness decreases to a certain point and then increases as the parameter a increases. With respect to kurtosis, for $\beta>2.5$, the behavior of kurtosis is similar to that of skewness. On the other hand, for $\beta<2.5$, the kurtosis increases to a certain point and then decreases as the additional parameter a increases.

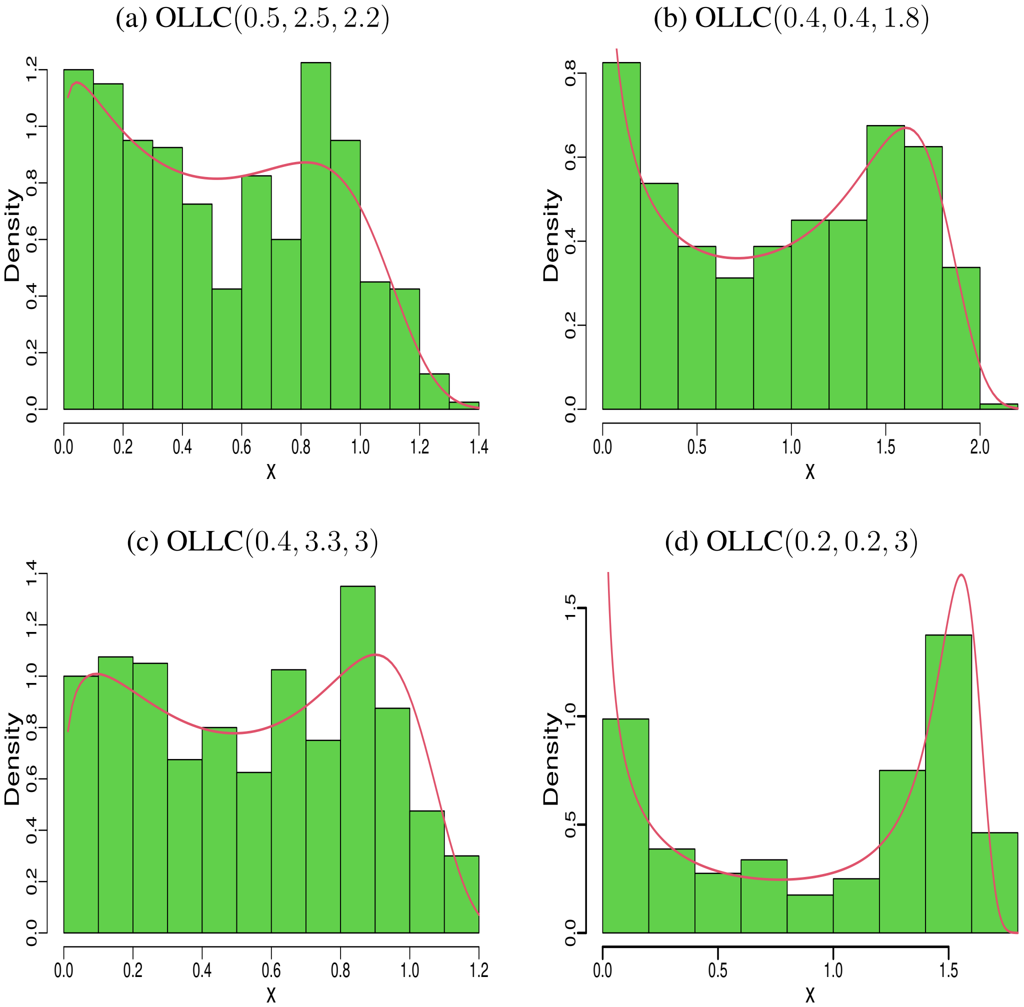

Figure 3: Simulated values of X for specific special cases.

By using expression (7), the rth moment of X is given by

$$\mathbb{E}[X^r] = \sum_{m=1}^{\infty} v_m \mathbb{E}[Y_m^r],$$

where $Y_m \sim \text{Chen}(m\lambda, \beta)$.

The rth moment of the random variable Y having density function (3) is given according to Pogány et al. (2017) as

$$\mathbb{E}[Y^r] = \lambda\, \mathrm{e}^{\lambda}\, \mathbb{D}_t^{r\beta-1} \left[\frac{\Gamma(t+1,\lambda)}{\lambda^{t+1}}\right]_{t=0}. \tag{10}$$

Here,

$$\mathbb{D}_t^{p} \left[\frac{\Gamma(t+1,\lambda)}{\lambda^{t+1}}\right]_{t=0} = \Gamma(p+1) \sum_{k\geq 0} \frac{(2)_k}{k!}\, \Phi_{\mu,1}^{(0,1)}(-k, p+1, 1)\ {}_1F_1(k+2; 2; -\lambda),$$

where $\Gamma(p) = \int_0^{\infty} u^{p-1}\mathrm{e}^{-u} du$ is the gamma function, $\Phi_{\mu,1}^{(0,1)}(-a, p+1, 1) = \sum_{n\geq 0} \frac{(-a)^n}{n!(n+1)^{p+1}}$ for $\mu \in \mathbb{C}$, ${}_1F_1(a; b; x) = \sum_{n\geq 0} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}$, for $x, a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus Z_0^-$, is the confluent hypergeomet-

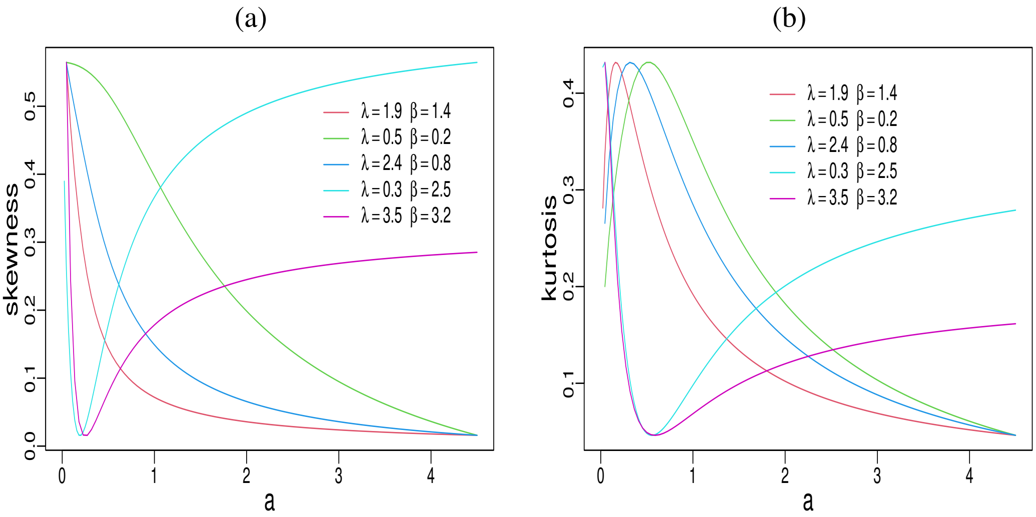

Figure 4: Skewness (a) and kurtosis (b) plots of X as function of a.

ric function and $(\lambda)_\eta = \frac{\Gamma(\lambda+\eta)}{\Gamma(\lambda)}$, for $\lambda \in \mathbb{C} \setminus \{0\}$, is the generalized Pochhammer symbol, under the convention $(0)_0 = 1$.

Using Equation (10), the rth moment of X reduces to

$$\mathbb{E}[X^r] = \lambda \sum_{m=1}^{\infty} v_m\, m\, \mathrm{e}^{m\lambda}\, \mathbb{D}_t^{r\beta-1} \left[\frac{\Gamma(t+1, m\lambda)}{(m\lambda)^{t+1}}\right]_{t=0}.$$

For $z > 0$, the rth incomplete moment of the random variable Y with Chen distribution, say $q_r(z; \lambda, \beta) = \int_0^z y^r\, g(y; \lambda, \beta) dy$, follows from Pogány et al. (2017) as

$$q_r(z; \lambda, \beta) = \lambda \mathrm{e}^{\lambda} \sum_{n,k \geq 0} \sum_{j=1}^{k} \frac{(2)_{n+k}}{(2)_n} \frac{(-1)^{n+j} \lambda^n \binom{k}{j}}{n!k!(j+1)^{r\beta^{-1}+1}} \gamma(r\beta^{-1}, (j+1)(1-z^{-1})), \tag{11}$$

where $\gamma(p, x) = \int_0^x u^{p-1} \mathrm{e}^{-u} du$ is the incomplete gamma function.

So, using Equations (7) and (11), the rth incomplete moment of X can written as

$$m_r(z) = \lambda \sum_{m=1}^{\infty} m \mathrm{e}^{m\lambda} v_m \sum_{n,k \geq 0} \sum_{j=1}^{k} \frac{(2)_{n+k}}{(2)_n} \frac{(-1)^{n+j} (m\lambda)^n \binom{k}{j}}{n!k!(j+1)^{r\beta^{-1}+1}} \gamma(r\beta^{-1}, (1-z^{-1})(j+1)).$$

The moment generating function (mgf) of $Y \sim \text{Chen}(\lambda, \beta)$, $M_Y(t) = \mathbb{E}[\mathrm{e}^{-tY}]$, $t > 0$, can be written, according to Pogány et al. (2017) by

$$M_Y(t) = \lambda \beta \mathrm{e}^{\lambda} t^{-\beta} \sum_{n \geq 0} \frac{(-\lambda)^n}{n!}\, {}_1\Psi_0 \left[(\beta, \beta); -; \frac{n+1}{t^\beta}\right], \tag{12}$$

where

$${}_1\Psi_0 \left[(a, b); -; z\right] = \sum_{n \geq 0} \frac{\Gamma(a+bn)\, z^n}{n!}, \qquad z, a \in \mathbb{C}, b > 0,$$

is the generalized Fox–Wright function.

Thus, using (7) and (12), the mgf of X follows as

$$M_X(t) = \lambda \beta t^{-\beta} \sum_{m=1}^{\infty} \sum_{n \geq 0} \frac{m \mathrm{e}^{m\lambda} (-m\lambda)^n v_m}{n!}\, {}_1\Psi_0 \left[(\beta, \beta); -; \frac{n+1}{t^\beta}\right].$$

4 Estimation

Consider $x_1, \cdots, x_n$ the observed values of $X_1, \cdots, X_n \sim \text{OLLC}(a, \lambda, \beta)$. Then, the log-likelihood for $(a, \lambda, \beta)^\top$ is given by

$$\begin{aligned}
\mathcal{L}(a,\lambda,\beta) =& n\log a\lambda\beta + n\lambda + (\beta-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n} x_i^{\beta} - \lambda\sum_{i=1}^{n} \mathrm{e}^{x_i^{\beta}} \\
&+ (a-1)\sum_{i=1}^{n}\log\left[\mathrm{e}^{\lambda(1-\mathrm{e}^{x_i^{\beta}})} - \mathrm{e}^{2\lambda(1-\mathrm{e}^{x_i^{\beta}})}\right] \\
&- 2\sum_{i=1}^{n}\log\left\{\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x_i^{\beta}})}\right]^{a} + \mathrm{e}^{a\lambda(1-\mathrm{e}^{x_i^{\beta}})}\right\}.
\end{aligned} \tag{13}$$

The components of the score vector $U(a, \lambda, \beta) = (U_a, U_\lambda, U_\beta)^\top$ of the log-likelihood (13) are given by

$$\begin{aligned}
U_a =& \frac{n}{a} + \sum_{i=1}^{n}\log\left[\mathrm{e}^{\lambda(1-\mathrm{e}^{x_i^{\beta}})} - \mathrm{e}^{2\lambda(1-\mathrm{e}^{x_i^{\beta}})}\right] - 2\sum_{i=1}^{n}\frac{t(x_i)^a\log t(x_i) + \lambda\mathrm{e}^{a\lambda(1-\mathrm{e}^{x_i^{\beta}})}(1-\mathrm{e}^{x_i^{\beta}})}{t(x_i)^a + \mathrm{e}^{a\lambda(1-\mathrm{e}^{x_i^{\beta}})}}, \\
U_\lambda =& n + \frac{n}{\lambda} - \sum_{i=1}^{n}\mathrm{e}^{x_i^{\beta}} + (a-1)\sum_{i=1}^{n}\frac{\mathrm{e}^{\lambda(1-\mathrm{e}^{x_i^{\beta}})}(1-\mathrm{e}^{x_i^{\beta}}) - 2\mathrm{e}^{2\lambda(1-\mathrm{e}^{x_i^{\beta}})}(1-\mathrm{e}^{x_i^{\beta}})}{\mathrm{e}^{\lambda(1-\mathrm{e}^{x_i^{\beta}})} - \mathrm{e}^{2\lambda(1-\mathrm{e}^{x_i^{\beta}})}} \\
&- 2\sum_{i=1}^{n}\frac{a\mathrm{e}^{a\lambda(1-\mathrm{e}^{x_i^{\beta}})}(1-\mathrm{e}^{x_i^{\beta}}) - at(x_i)^{a-1}\mathrm{e}^{\lambda(1-\mathrm{e}^{x_i^{\beta}})}(1-\mathrm{e}^{x_i^{\beta}})}{t(x_i)^a + \mathrm{e}^{a\lambda(1-\mathrm{e}^{x_i^{\beta}})}}, \\
U_\beta =& \frac{n}{\beta} + \sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n} x_i^{\beta}\log x_i - \lambda\sum_{i=1}^{n} x_i^{\beta}\mathrm{e}^{x_i^{\beta}}\log x_i \\
&+ (a-1)\sum_{i=1}^{n}\frac{2\lambda x_i^{\beta}\mathrm{e}^{2\lambda(1-\mathrm{e}^{x_i^{\beta}})+x_i^{\beta}}\log x_i - \lambda x_i^{\beta}\mathrm{e}^{\lambda(1-\mathrm{e}^{x_i^{\beta}})+x_i^{\beta}}\log x_i}{\mathrm{e}^{\lambda(1-\mathrm{e}^{x_i^{\beta}})} - \mathrm{e}^{2\lambda(1-\mathrm{e}^{x_i^{\beta}})}} \\
&- 2\sum_{i=1}^{n}\frac{a\lambda x_i^{\beta}t(x_i)^{a-1}\mathrm{e}^{\lambda(1-\mathrm{e}^{x_i^{\beta}})+x_i^{\beta}}\log x_i - a\lambda x_i^{\beta}\mathrm{e}^{a\lambda(1-\mathrm{e}^{x_i^{\beta}})+x_i^{\beta}}\log x_i}{t(x_i)^a + \mathrm{e}^{a\lambda(1-\mathrm{e}^{x_i^{\beta}})}},
\end{aligned}$$

where $t(x_i) = 1 - \exp\{\lambda(1 - \mathrm{e}^{x_i^{\beta}})\}$.

The maximum likelihood estimates (MLEs) $(\hat{b}, \hat{\lambda}, \hat{\beta})$ of (b, λ, β) are the simultaneous solutions of $U_a = U_\lambda = U_\beta = 0$. These solutions are those $(\hat{b}, \hat{\lambda}, \hat{\beta})$ values that maximize the log-likelihood (13). These MLEs can not be obtained analytically. So, the use of interactive methods such as the quasi-Newton BFGS and Newton-Raphson algorithms is required.

4.1 Simulation

Here, a Monte Carlo simulation is performed to evaluate the accuracy of the MLEs for the OLLC model. The `optim` function available in R Project (R Core Team, 2020) is used to obtain these MLEs. The random number generation is done using the Equation (9). The simulation of Monte Carlo is performed with $1,000$ repetitions and with samples sizes of $n = \{100, 200, 300, 400\}$ for two scenarios. The true parameter values are: $a = 0.5$, $\lambda = 1.7$

and $\beta = 1.9$ for scenario 1 and $a = 4.1$, $\lambda = 0.5$ and $\beta = 0.6$ for scenario 2. The evaluation is based on the average estimates (AEs), biases and mean square errors (MSEs).

The results of the simulations are described in Table 2. Note that in both scenarios, when n grows the MLEs converge to the true parameters and the biases and MSEs decrease. Thus, the MLEs for the OLLC model is in agreement with what is expected from asymptotic theory.

Table 2: AEs, biases and MSEs for the OLLC distribution under scenarios 1 to 2.

scenario 1: $(a, \lambda, \beta) = (0.5, 1.7, 1.9)$						
Par	$n = 100$			$n = 200$		
	AE	Bias	MSE	AE	Bias	MSE
a	0.49334	-0.00666	0.00842	0.49808	-0.00192	0.00422
λ	1.76273	0.06273	0.09419	1.72808	0.02808	0.04383
β	1.99142	0.09142	0.09906	1.94346	0.04346	0.04598
Par	$n = 300$			$n = 400$		
	AE	Bias	MSE	AE	Bias	MSE
a	0.50032	0.00032	0.00277	0.49993	-0.00007	0.00196
λ	1.71917	0.01917	0.02692	1.71208	0.01208	0.01998
β	1.91988	0.01988	0.02667	1.91904	0.01904	0.01982
scenario 2: $(a, \lambda, \beta) = (4.1, 0.5, 0.6)$						
Par	$n = 100$			$n = 200$		
	AE	Bias	MSE	AE	Bias	MSE
a	4.40437	0.30437	9.00016	4.58201	0.48201	6.46828
λ	0.52494	0.02494	0.00460	0.50940	0.00940	0.00207
β	0.74695	0.14695	0.14609	0.65482	0.05482	0.06959
Par	$n = 300$			$n = 400$		
	AE	Bias	MSE	AE	Bias	MSE
a	4.51684	0.41684	4.28313	4.36601	0.26601	2.49295
λ	0.50459	0.00459	0.00123	0.50339	0.00339	0.00088
β	0.62444	0.02444	0.04225	0.61910	0.01910	0.02958

5 Regression model

Taking $\nu = \text{median}(X)$, the parameter λ can be written as

$$\lambda = \frac{\log(0.5)}{1 - \mathrm{e}^{\nu^\beta}}.$$

By this parameterization, the pdf of the reparameterized OLLC (ROLLC) distribution is given by

$$f_{\text{OLLC}}(x; a, \beta, \nu) = \frac{a\beta\log(0.5)x^{\beta-1}\mathrm{e}^{x^\beta + \log(0.5)\frac{1-\mathrm{e}^{x^\beta}}{1-\mathrm{e}^{\nu^\beta}}}\left[\mathrm{e}^{\log(0.5)\frac{1-\mathrm{e}^{x^\beta}}{1-\mathrm{e}^{\nu^\beta}}} - \mathrm{e}^{2\log(0.5)\frac{1-\mathrm{e}^{x^\beta}}{1-\mathrm{e}^{\nu^\beta}}}\right]^{a-1}}{(1-\mathrm{e}^{\nu^\beta})\left\{\left[1 - \mathrm{e}^{\log(0.5)\frac{1-\mathrm{e}^{x^\beta}}{1-\mathrm{e}^{\nu^\beta}}}\right]^a + \mathrm{e}^{a\log(0.5)\frac{1-\mathrm{e}^{x^\beta}}{1-\mathrm{e}^{\nu^\beta}}}\right\}^2},$$

with corresponding survival function

$$S_{\text{OLLC}}(x;a,\beta,\nu)=\frac{\mathrm{e}^{a\log(0.5)\frac{1-\mathrm{e}^{x^\beta}}{1-\mathrm{e}^{\nu^\beta}}}}{\left[1-\mathrm{e}^{\log(0.5)\frac{1-\mathrm{e}^{x^\beta}}{1-\mathrm{e}^{\nu^\beta}}}\right]^a+\mathrm{e}^{a\log(0.5)\frac{1-\mathrm{e}^{x^\beta}}{1-\mathrm{e}^{\nu^\beta}}}},$$

where $\nu>0$ denotes the median of the distribution. Here, the random variable under parameterization in the median is denoted as $X\sim$ ROLLC(a,β,ν). For $a=1$, the reparameterized Chen (RC) distribution is obtained.

Since ν denotes the median of X, then this median can be modeled by covariates. The regression model for the median ν_i of x_i is given by

$$g(\nu_i)=\sum_{m=1}^{k}w_{im}\delta_m=\eta_i, \tag{14}$$

in which $\boldsymbol{\delta}=(\delta_1,\ldots,\delta_k)^\top$ is k-vector of unknown parameters, $w_{i1},\ldots,w_{ik}$ are observations on k covariates $(k<n)$, which are assumed fixed and known and η_i is the linear predictor. Here, $g(\cdot)$ is strictly monotonic and twice differentiable link function, such that $g:\mathbb{R}^+\to\mathbb{R}$. Examples of the link functions can be: logarithmic function $g(\nu)=\log\nu$ and square root function $g(\nu)=\sqrt{\nu}$.

In survival data analysis, the data are often censored. This happens in situations where the patient has left follow-up, died of another cause or simply the study has ended. In situations like this, the time to failure is longer than the observed time. However, these censored observations cannot be ignored. Thus, methods to model this censoring must be realized.

Suppose that individuals have a lifetime X_i and right censoring time $X_i^{(c)}$, where X_i and $X_i^{(c)}$ are independents. Under censoring, the observed data set are (T_i,ψ_i), where $T_i=\min\{X_i,X_i^{(c)}\}$ and ψ_i is a censoring indicator variable defined by

$$\psi_i=\begin{cases}1, & X_i\le X_i^{(c)},\\ 0, & X_i>X_i^{(c)}.\end{cases}$$

Let the independent random variables $X_i\sim$ ROLLC(a,β,ν_i), $i=1,\ldots,n$, with observed values x_i. From pdf $f_{\text{OLLC}}(x;a,\beta,\nu)$ and survival function $S_{\text{OLLC}}(x;a,\beta,\nu)$, the log-likelihood under censoring, for ROLLC regression model (14) is given by

$$\mathcal{L}(a,\beta,\boldsymbol{\delta})=\sum_{i=n}^{n}\psi_i\mathcal{L}_i(a,\beta,\nu_i)+\sum_{i=1}^{n}(1-\psi_i)\mathcal{L}_i^{(c)}(a,\beta,\nu_i),$$

where

$$\begin{aligned}\mathcal{L}_i(a,\beta,\nu_i)=&\log(a\beta)+\log\left(\frac{\log(0.5)}{1-\mathrm{e}^{\nu_i^\beta}}\right)+(\beta-1)\log x_i+x_i^\beta+\log(0.5)\frac{1-\mathrm{e}^{x_i^\beta}}{1-\mathrm{e}^{\nu_i^\beta}}\\&+(a-1)\log\left[\mathrm{e}^{\log(0.5)\frac{1-\mathrm{e}^{x_i^\beta}}{1-\mathrm{e}^{\nu_i^\beta}}}-\mathrm{e}^{2\log(0.5)\frac{1-\mathrm{e}^{x_i^\beta}}{1-\mathrm{e}^{\nu_i^\beta}}}\right]\\&-2\log\left\{\left[1-\mathrm{e}^{\log(0.5)\frac{1-\mathrm{e}^{x_i^\beta}}{1-\mathrm{e}^{\nu_i^\beta}}}\right]^a+\mathrm{e}^{a\log(0.5)\frac{1-\mathrm{e}^{x_i^\beta}}{1-\mathrm{e}^{\nu_i^\beta}}}\right\}\end{aligned}$$

and

$$\mathcal{L}_i^{(c)}(a,\beta,\nu_i) = a\log(0.5)\frac{1-\mathrm{e}^{x_i^\beta}}{1-\mathrm{e}^{\nu_i^\beta}} - \log\left\{\left[1-\mathrm{e}^{\log(0.5)\frac{1-\mathrm{e}^{x_i^\beta}}{1-\mathrm{e}^{\nu_i^\beta}}}\right]^a + \mathrm{e}^{a\log(0.5)\frac{1-\mathrm{e}^{x_i^\beta}}{1-\mathrm{e}^{\nu_i^\beta}}}\right\}.$$

The MLEs of $(a,\beta,\boldsymbol{\delta})$, says $(\hat{a},\hat{\beta},\hat{\boldsymbol{\delta}})$, can be found by maximizing $\mathcal{L}(a,\beta,\boldsymbol{\delta})$ numerically with respect to its parameters. This maximization can be performed without difficulty in some statistical packages, such as R (R Core Team, 2020) (`optim` function) and Ox (Doornik, 2018) (sub-routine `MaxBFGS`).

The observed information matrix $J(\hat{a},\hat{\beta},\hat{\boldsymbol{\delta}})$ can be calculated numerically. Under general conditions of regularity, the multivariate normal $N_{k+2}(0, J(\hat{a},\hat{\beta},\hat{\boldsymbol{\delta}})^{-1})$ distribution can be used to obtain the standard errors of the estimates and their confidence intervals.

Since RC distribution is a special case of ROLLC when $a=1$, the likelihood ratio (LR) between these two regressions models can be used. The null hypothesis and alternative hypothesis of the LR test are $\mathcal{H}_0 : a = 0$ and $\mathcal{H}_1 : a \neq 0$, respectively. The LR test is given by

$$w = 2\left[\mathcal{L}(\hat{a},\hat{\beta},\hat{\boldsymbol{\delta}}) - \mathcal{L}(1,\tilde{\beta},\tilde{\boldsymbol{\delta}})\right],$$

where $(\hat{a},\hat{\beta},\hat{\boldsymbol{\delta}})$ are the MLEs under $\mathcal{H}_1$ and $(\tilde{a},\tilde{\beta},\tilde{\boldsymbol{\delta}})$ are the MLEs under $\mathcal{H}_0$. For n large and under $\mathcal{H}_0$, $w \sim \chi_1^2$, where χ_1^2 is the chi-square distribution with one degree-freedom. The null hypothesis is rejected when $w > \chi_1^2(1-\alpha)$, where $\chi_1^2(1-\alpha)$ is the quantile $(1-\alpha)$ of χ_1^2.

5.1 Simulation

To show the accuracy of the maximum likelihood estimators for the LL regression model, under censoring, Monte Carlo simulations with 10000 replicates are performed. The censoring percentages (cp) is of 0%, 10% and 30% for the sample sizes $n = \{60, 120, 300\}$. The evaluation is based on the AEs and the MSEs.

The simulated model is given by

$$\log\nu_i = \delta_1 + \delta_2 w_{i2} + \delta_3 w_{i3} + \delta_4 w_{i4}, \quad i,\ldots,n. \tag{15}$$

The covariates were generated from the standard uniform distribution, i.e., $w_{im} \sim \mathcal{U}(0,1)$, $m = 2,3,4$. The true parameters adopted are: $\delta_1 = 1.5$, $\delta_2 = 0.4$, $\delta_3 = -2.6$ and $\phi = 3.2$. The response variables $x_1,\ldots,x_n$ are generated from Equation (9) with $\lambda_i = \log(0.5)/[1-\exp(\nu_i^\beta)]$, where ν_i is obtained from structure of the regression model (15). The censoring times, $x_1^{(c)},\ldots,x_n^{(c)}$, are generated from $x_i^{(c)} \sim \mathcal{U}(0,\theta_i)$, where θ_i is such that it satisfies $\Pr(X_i > \theta_i) = \mathrm{cp}$, with $\mathrm{cp} \in \{0.0, 0.15, 0.30\}$. The lifetimes considered in each fit are given as $t_i = \min\{x_i, x_i^{(c)}\}$ with censoring indicator

$$\psi_i = \begin{cases} 1, & x_i \leq x_i^{(c)}, \\ 0, & x_i > x_i^{(c)}. \end{cases}$$

Table 3 shows the AEs and MSEs ot the simulation for the ROLLC regression model under censoring. Note that, for all censoring levels, when n increases, the MLEs converge to true parameters and MSEs decrease. These results show the consistency of the MLEs of the ROLLC regression model.

Table 3: Monte Carlo simulation results for ROLLC regression model.

n	Par	cp $= 0.0$		cp $= 0.15$		cp $= 0.30$	
		AE	MSE	AE	MSE	AE	MSE
50	a	0.46012	0.02228	1.33581	2.27373	1.53274	2.79531
	β	2.14037	0.35895	1.22666	1.14958	0.97721	1.31206
	δ_1	0.50981	0.10719	0.48935	0.50827	0.50680	0.57947
	δ_2	−2.37359	0.13762	−2.42061	0.67938	−2.37291	0.83574
	δ_3	−1.58785	0.13869	−1.62905	0.64295	−1.63978	0.81537
	δ_4	0.69120	0.12593	0.68888	0.59971	0.67306	0.67834
120	a	0.48252	0.00739	0.85710	0.63530	1.15040	1.53629
	β	1.93315	0.09275	1.49941	0.67436	1.23142	0.94895
	δ_1	0.49586	0.02819	0.51191	0.16418	0.52667	0.23092
	δ_2	−2.37469	0.03889	−2.40705	0.19022	−2.40490	0.27629
	δ_3	−1.59224	0.03313	−1.64265	0.19940	−1.63890	0.28104
	δ_4	0.69789	0.03860	0.68693	0.20312	0.68282	0.29667
300	a	0.49157	0.00254	0.66579	0.22113	0.80176	0.35226
	β	1.85490	0.02447	1.61388	0.36209	1.41223	0.58115
	δ_1	0.50343	0.01009	0.49475	0.04988	0.50544	0.07188
	δ_2	−2.39489	0.01288	−2.41209	0.07038	−2.40722	0.09842
	δ_3	−1.59495	0.01164	−1.59973	0.06040	−1.61254	0.08987
	δ_4	0.69555	0.01223	0.69908	0.06332	0.70345	0.09184

6 Applications

In this section, applications to real censored and uncensored data are considered to show the potentiality of the proposed model.

6.1 Uncensored data

Three data sets are considered, namely:

1. The first one refers to graft survival times (in months) of 148 renal transplant patients (graft data). This data also was analyzed by (Kayal et al., 2019).
2. The second one is the famous Aarset data, which refers to the lifetimes of 50 devices (Aarset data). Mudholkar and Srivastava (1993) also used this data.
3. The third one ($n = 150$) referes to petal width (in cm) samples of three species of Iris (Iris setosa, Iris virginica and Iris versicolor) (Iris data). These data were introduced by Fisher (1936) and are available in software R (R Core Team, 2020).

The OLLC distribution is compared with three others known distributions, namely: Burr XII (BXII), exponentiated-Weibull (EW) (Mudholkar and Srivastava, 1993) and gamma-Lomax (GL) (Cordeiro et al., 2015) distributions. The pdfs of the BXII, EW and GL distributions are given by

$$f_{\text{BXII}}(x; s, d, c) = \frac{cd}{s^c} x^{c-1}[1 + (x/s)^c]^{-(d+1)}, \quad x > 0,$$

$$f_{\text{EW}}(x; a, \beta, \alpha) = a\alpha\beta^\alpha x^{\alpha-1} \exp\{-(\beta x)^\alpha\} [1 - \exp\{-(\beta x)^\alpha\}]^{a-1}, \quad x > 0$$

and

$$f_{\mathrm{GL}}(x; a, \beta, \alpha) = \frac{\alpha\beta^{\alpha}}{\Gamma(a)}[\beta + x]^{-(\alpha+1)}\{-\alpha \log[\beta/(\beta + x)]\}^{a-1}, \quad x > 0,$$

respectively, where $s, d, c, a, \beta, \alpha > 0$.

To choose the best model, the Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics described in Chen and Balakrishnan (1995) are adopted. The best model is the one with the lowest values of these statistics.

Tables 4, 5 and 6 present the MLEs with the standard errors (SEs) in parentheses and the information criteria for from graft, Aarset and Iris datasets, respectively. In the three datasets, the statistics W^* and A^* point to the OLLC distribution as the best model.

Graphical analysis is also an important indicator for choosing a model. Figures 5, 6 and 7 present the estimated pdfs and cdfs for the graft, Aarset and Iris datasets, respectively. It is observed in these figures that, in the three datasets, the OLLC model has a better fit, corroborating with the W^* and A^* statistics.

Table 4: Estimation results for graft data.

Model	Estimate			W^*	A^*
OLLC(a, λ, β)	0.7488	0.0266	0.4324	**0.0904**	**0.7214**
	(0.1089)	(0.0093)	(0.0258)		
BXII(s, d, c)	194.0200	12.9412	1.0495	0.5779	3.5751
	(101.8254)	(6.1792)	(0.0746)		
EW(a, β, α)	0.1427	0.0250	4.6904	0.1231	0.8002
	(0.0124)	(0.0012)	(0.0136)		
GL(a, β, α)	0.9389	232.4819	13.2583	0.6001	3.6987
	(0.0963)	(111.8847)	(6.1279)		

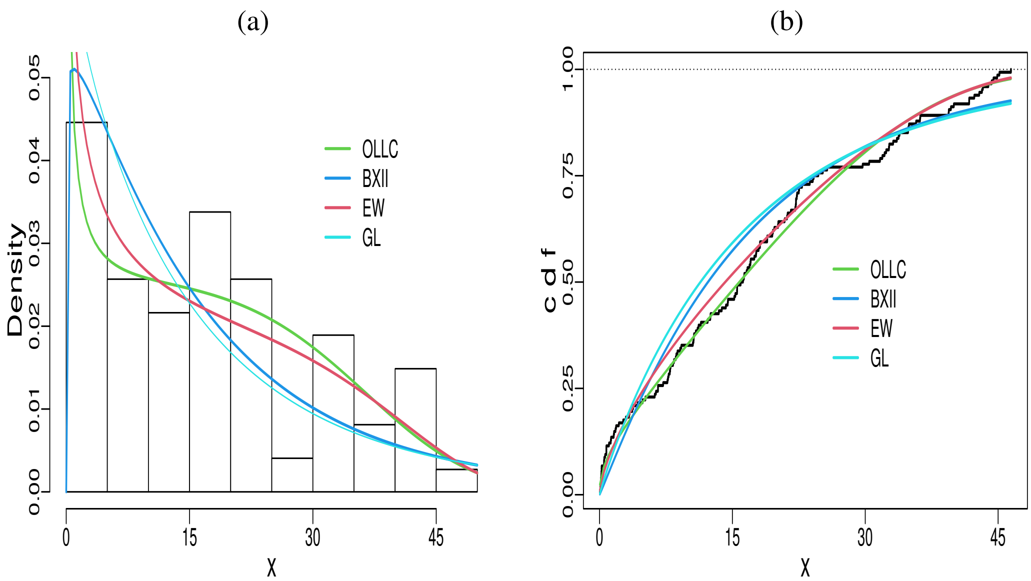

Figure 5: Estimated pdfs (a) and estimated cdfs (b) for graft data.

Table 5: Estimation results for Aarset data.

Model	Estimate			W^*	A^*
OLLC(a,λ,β)	0.3762	0.0023	0.4658	**0.1660**	**1.2675**
	(0.0714)	(0.0006)	(0.0108)		
BXII(s,d,c)	127.7912	3.6590	1.0432	0.5545	3.3136
	(61.0142)	(1.6482)	(0.1242)		
EW(a,β,α)	0.1456	0.0109	4.6957	0.2732	1.8064
	(0.0218)	(0.0009)	(0.0228)		
GL(a,β,α)	0.8407	128.1478	2.8494	0.5668	3.3768
	(0.1506)	(58.8085)	(1.1265)		

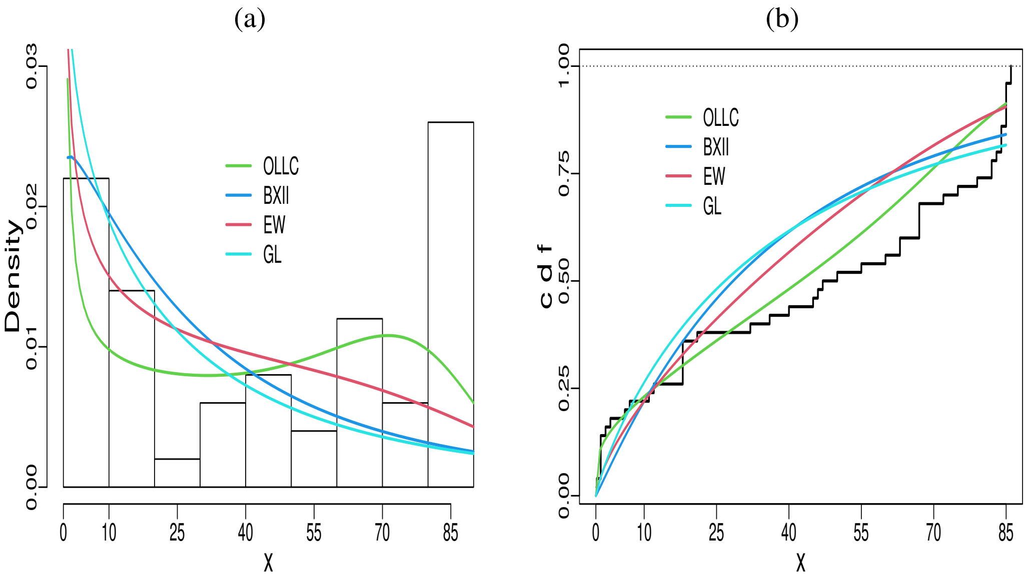

Figure 6: Estimated pdfs (a) and estimated cdfs (b) for Aarset data.

Table 6: Estimation results for Iris data.

Model	Estimate			W^*	A^*
OLLC(a,λ,β)	0.5485	0.2116	1.4206	0.3713	2.8008
	(0.0636)	(0.0450)	(0.0981)		
BXII(s,d,c)	17.4015	44.3020	1.4630	1.3964	8.1928
	(11.0617)	(38.6172)	(0.1029)		
EW(a,β,α)	0.1008	0.4287	9.6595	0.5545	3.6607
	(0.0085)	(0.0101)	(0.0224)		
GL(a,β,α)	1.5790	40.9046	54.9308	1.5931	9.1745
	(0.1678)	(33.3069)	(43.6705)		

6.2 Censored data

The data considered refer to a study, described in Klein and Moeschberger (1997), of 90 male patients diagnosed in the 1970-1978 period with laryngeal cancer who were followed up until 01/01/1983. This study has the presence of two exogenous variables: w_1 and w_2. The variable w_1

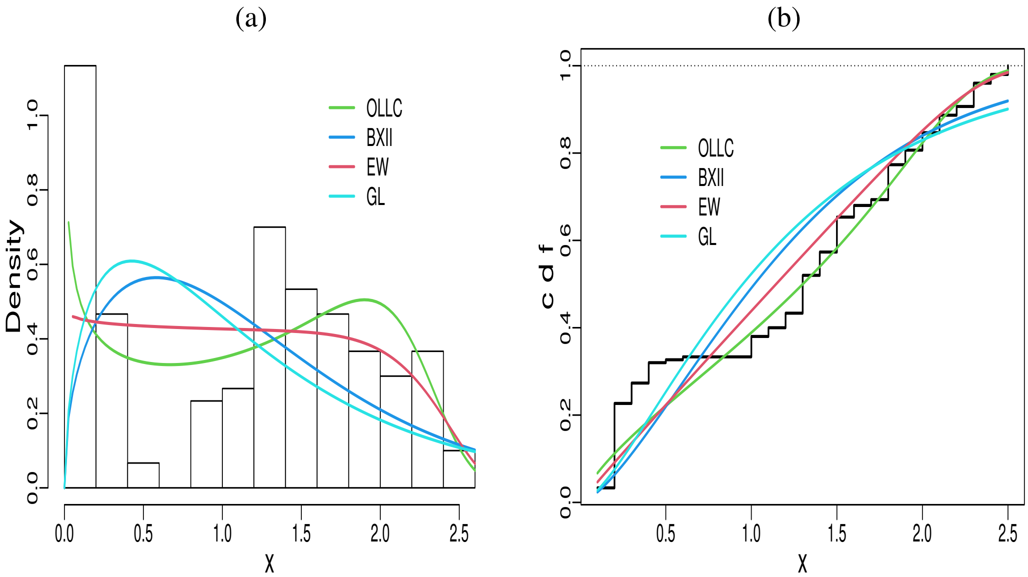

Figure 7: Estimated pdfs (a) and estimated cdfs (b) for Iris data.

refers to the age of each patient (in years). Alrady, the variable w_2 is a dummy variable denoting the stage of the disease ($1 =$ primary tumor, $2 =$ nodule involvement, $3 =$ metastasis, and $4 =$ combinations of the three previous stages). The response variable (x) represents the respective failure or censoring times (in months). The censoring type is of failure ($0 =$ censoring and $1 =$ lifetime observed) and the percentage of censoring is 44.44%.

The ROLLC regression to be estimated is given by

$$\log \nu_i = \delta_1 + \delta_2 w_{i1} + \delta_3 D_{i2} + \delta_4 D_{i3} + \delta_5 D_{i4}, \quad i = 1, \ldots, 90,$$

where ν_i denotes the median and $D_{i2} = 1\ (w_{i2} = 2)$, $D_{i3} = 1\ (w_{i2} = 3)$, $D_{i4} = 1\ (w_{i2} = 4)$.

Table 7 shows the estimates results for the ROLLC and RC models. In both models, the coefficients δ_2 and δ_3 were not statistically significant. The Akaike Information Criterion (AIC) and Hannan–Quinn Information Criterion (HQIC) point to the ROLLC model, while the Bayesian Information Criterion (BIC) points to the RC model. The LR test to discriminate between ROLLC and RC models is $w = 3.1829$ and the critical value at the 8% significance level is $\chi^2_1(0.92) = 3.0649$. Thus, the null hypothesis of the RC model is rejected, and the ROLLC model is a better fit for the data.

7 Conclusions

A new distribution that extends the Chen distribution has been proposed. This new model adds an extra shape parameter to the Chen distribution, giving more flexibility to the shapes of the Chen distribution's density curves and failure rates. The density of the new distribution admits several forms, including the bimodality. Regarding the failure rate function, it can be decreasing, bathtub, unimodal, unimodal-bathtub. By inversion method, we show that random numbers of the new distribution can be performed easily.

From this new distribution a regression model for censored data is proposed. This model has a regression structure at the median, with an aid of a link function.

Table 7: Summary of the estimates of the ROLLC and RC regression models.

Par	Estimate	Std. Error	z-value	p-value
		ROLLC		
a	5.80621	14.75336	0.39355	0.69391
β	0.10963	0.27546	0.39799	0.69063
δ_1	3.02360	0.91680	3.29800	0.00097
δ_2	-0.01449	0.01312	-1.10461	0.26933
δ_3	-0.11154	0.38699	-0.28822	0.77318
δ_4	-0.72458	0.38552	-1.87949	0.06018
δ_5	-1.69759	0.43989	-3.85913	0.00011
	AIC	BIC	HQIC	
	297.00966	314.50833	304.06615	
		RC		
β	0.51890	0.04341	11.95351	0.00000
δ_1	2.96285	0.70027	4.23100	0.00002
δ_2	-0.01452	0.01030	-1.40874	0.15891
δ_3	-0.08895	0.30282	-0.29373	0.76896
δ_4	-0.44868	0.25453	-1.76280	0.07793
δ_5	-1.53309	0.37696	-4.06700	0.00005
	AIC	BIC	HQIC	
	298.19265	313.19151	304.24107	

The estimation of unknown parameters is performed by the maximum likelihood method. Monte Carlo simulations were performed, showing the accuracy of the maximum likelihood estimators for the proposed model. The usefulness of the proposed model in practice is shown through three applications to uncensored data and one application to censored data.

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